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HIGH ORDER CONTINUITY IMPLIES GOOD APPROXIMATIONS TO SOLUTIONS --ETC(U)
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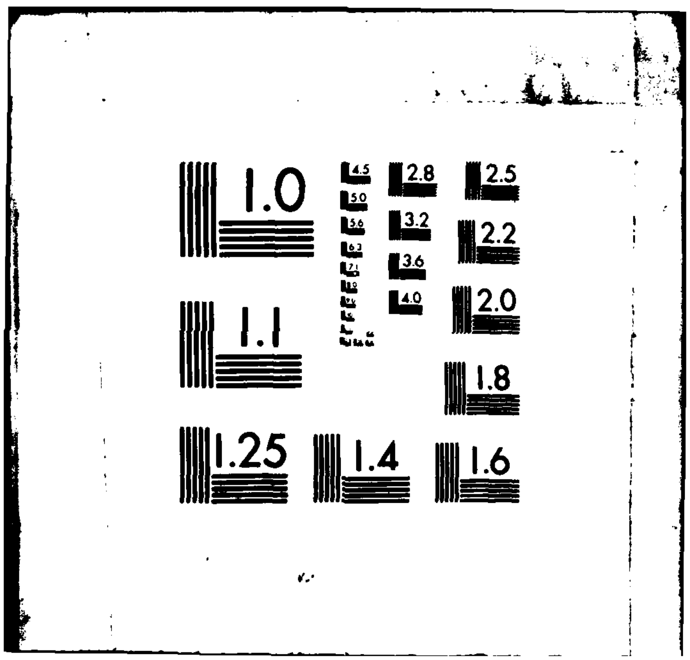
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HIGH ORDER CONTINUITY IMPLIES GOOD APPROXIMATIONS TO SOLUTIONS OF CERTAIN FUNCTIONAL EQUATIONS

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ABSTRACT

Our title describes a phenomenon best illustrated by our Theorem 1.

1. Let $h(x)$ be an entire function of exponential type $A < 2\pi$. We show that there is a unique entire function $f(x)$ of exponential type A satisfying the functional equation

$$f(x+1) - f(x) = h(x), \quad \text{with } f(0) = 0 \ . \tag{1}$$

2. We define a function $S_n(x)$ which reduces to a polynomial in $[0,1)$:

$$S_n(x) = a_1x + a_2x^2 + \dots + a_nx^n \quad \text{if} \quad 0 \le x < 1 \ , \tag{2}$$

and we extend the definition of $S_n(x)$ to all real x by asking that it satisfy

$$S_n(x+1) - S_n(x) = h(x) \quad \text{for all real } x. \tag{3}$$

If we require also that

$$S_n^{(\nu)}(+0) = S_n^{(\nu)}(-0) \quad \text{for} \quad \nu = 0,1,\dots,n-1 \tag{4}$$

then we have

$$S_n(x) \in C^{n-1}(\mathbb{R}) \tag{5}$$

and also

$$\lim_{n\to\infty} S_n(x) = f(x) \quad \text{uniformly for all real } x \ . \tag{6}$$

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Our title stresses the phenomenon that the C^{n-1}-continuity requirement (5) implies the approximation property (6).

As a second example we deal with the functional equation

$$f(x) = xf(x+1), \quad \text{with} \quad f(1) = 1 \quad , \tag{7}$$

satisfied by $f(x) = 1/\Gamma(x)$.

AMS (MOS) Subject Classifications: 15A06, 39B20, 41A10

Key Words: Functional equations, Approximations

Work Unit Number 3 - Numerical Analysis and Computer Science

SIGNIFICANCE AND EXPLANATION

Examples are given of functional equations, like $f(x+1) - f(x) = h(x)$, where a solution $f(x)$ is everywhere defined, provided that $f(x)$ is prescribed in the interval $0 \leq x < 1$. We define a function $S_n(x)$ by setting $S_n(x) = P_n(x)$ in $0 \leq x < 1$, where $P_n(x)$ is an as yet unknown polynomial of degree n, and requiring $S_n(x)$ to satisfy our functional equation for all real x. We now impose the

Continuity Requirement. We require the composite function $S_n(x)$ to have $n - 1$ continuous derivatives at the point $x = 0$.

It is shown in our examples that the Continuity Requirement has the following consequences.

1°, $S_n(x)$ has at least $n - 2$ continuous derivatives for all real x.

2°. As $n \to \infty$ $S_n(x)$ converges to a solution $f(x)$ of our functional equation.

The fact that the Continuity Requirement implies the approximation property 2° explains the meaning of our title.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

HIGH ORDER CONTINUITY IMPLIES GOOD APPROXIMATIONS
TO SOLUTIONS OF CERTAIN FUNCTIONAL EQUATIONS

T. N. T. Goodman, I. J. Schoenberg and A. Sharma

1. Introduction and main results. We describe here an apparently new principle of generating polynomial approximations in $0 \leq x \leq 1$ to the solutions of certain functional equations. pg 1

Our mysterious title will become less so if we mention first the example of the exponential Euler spline $S_n(x;t)$ satisfying the functional equation

(1.1) $$S_n(x+1;t) = tS_n(x;t), \text{ with } S_n(0;t) = 1 \quad .$$

Here t is a constant such that

(1.2) $$t = |t|e^{i\alpha}, -\pi < \alpha < \pi, t \neq 0, t \neq 1 \quad .$$

Let

(1.3) $$P_n(x) = 1 + c_1x + c_2x^2 + \ldots + c_nx^n$$

be an as yet unspecified polynomial. We define the function $S_n(x;t)$ by requiring firstly, that

(1.4) $$S_n(x;t) = P_n(x) \text{ if } 0 \leq x < 1 \quad ,$$

secondly, that $S_n(x;t)$ should satisfy the functional equation (1.1) for all real x. The result so far is that $S_n(x;t)$ is a piecewise polynomial function depending on the n parameters $c_1,\ldots,c_n$.

Now comes the essential requirement: We ask that

(1.5) $$S_n(x;t) \in C^{n-1}(\mathbb{R}) \quad .$$

This is what turns $S_n(x;t)$ into the exponential Euler spline. In fact it suffices to require C^{n-1}-continuity only near the point $x = 0$, because (1.1) will propagate this order of continuity to all other integer points.

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Concerning the continuity at $x = 0$: By (1.4) and (1.1) we see that in $0 \leqq x < 1$ we have $S_n(x;t) = S_n(x) = P_n(x)$, while in $-1 \leqq x < 0$ we have $S_n(x) = S_n(x+1)/t = P_n(x+1)/t$. These two functions $P_n(x)$ and $P_n(x+1)/t$ will join at $x = 0$ with $n - 1$ continuous derivatives iff $P_n^{(\nu)}(0) = P_n^{(\nu)}(1)/t$ or

(1.6) $$P_n^{(\nu)}(1) = tP_n^{(\nu)}(0), \quad (\nu = 0,\ldots,n-1) \quad .$$

These are precisely the equations that define the polynomial

(1.7) $$P_n(x) = A_n(x;t)/A_n(0;t) \quad ,$$

where $A_n(x;t)$ is defined by Euler's generating function

(1.8) $$\frac{t-1}{t-e^z} e^{xz} = \sum_0^\infty \frac{A_n(x;t)}{n!} z^n \quad .$$

For details see [9], where the exponential Euler splines were first introduced. It was also shown there that

(1.9) $$\lim_{n\to\infty} S_n(x;t) = t^x = |t|^x e^{i\alpha x} \quad \text{for all real} \quad x \quad .$$

To summarize: <u>We assume</u> (1.1) <u>and</u> (1.4); <u>now the continuity condition</u> (1.5) <u>implies the approximation property expressed by the limit relation</u> (1.9).

That continuity under similar circumstances implies approximation is not an isolated phenomenon. <u>Our second example is the functional equation</u>

(1.10) $$f(x+1) - f(x) = h(x), \quad \underline{\text{with}} \quad f(0) = 0 \quad ,$$

satisfied by the "sum" $f(x)$ of a prescribed function $h(x)$. We assume $h(x)$ to be entire of exponential type.

Our main result concerning (1.10) is

<u>Theorem 1.</u> 1. <u>We assume that</u>

(1.11) $|h^{(\nu)}(0)| < C\cdot A^\nu$, $(\nu = 0,1,\ldots)$, <u>where</u> C <u>and</u> A <u>are constants such that</u>

(1.12) $$A < 2\pi \quad .$$

The equation (1.10) *has a unique entire solution* $f(x)$ *such that*

(1.13) $$|f^{(\nu)}(0)| \leqq K\cdot A^{\nu}, \ (\nu = 0,1,\dots), \ \text{with } K \text{ constant} .$$

2. *We define* $S_n(x)$ *by setting*

(1.14) $$S_n(x) = \frac{1}{1!} a_1 x + \frac{1}{2!} a_2 x^2 + \dots + \frac{1}{n!} a_n x^n \ \text{ if } \ 0 \leqq x < 1 .$$

and extend its definition to all real x *so as to satisfy*

(1.15) $$S_n(x+1) - S_n(x) = h(x) .$$

If we require that

(1.16) $$S_n(x) \ \text{is of class} \ C^{n-1} \ \text{in a neighborhood of} \ x = 0 ,$$

then

(1.17) $$S_n(x) \in C^{n-1}(\mathbb{R}) .$$

Finally

(1.18) $$\lim_{n\to\infty} S_n(x) = f(x) \ \text{uniformly on} \ \mathbb{R} .$$

For a proof see §2. Theorem 1 is very close to a result of J. M. Whittaker [12, Theorem 3 on page 22]. Our proof brings out more clearly the main idea of this paper: That (1.14), (1.15), and (1.17) imply (1.18).

If e.g. $h(x) = 2^x$ (satisfying (1.11) with $A = \log 2 < 2\pi$), then the unique solution of (1.10) satisfying (1.13) is, of course, $f(x) = 2^x$. The approximation $S_n(x)$ to 2^x, furnished by Theorem 1, is different from the exponential Euler spline $S_n(x;t)$, for $t = 2$.

Our third and last example is perhaps the most interesting one: We consider the functional equation

(1.19) $$f(x) = xf(x+1), \ \text{with} \ f(1) = 1 ,$$

satisfied by the reciprocal Γ*-function*

(1.20) $$f(x) = \frac{1}{\Gamma(x)} .$$

Notice that (1.19) implies $f'(x) = f(x+1) + xf'(x+1)$, and setting $x = 0$ we obtain that

(1.21) $$f'(0) = f(1) = 1 .$$

Also (1.19) implies for $x = 0$ that $f(0) = 0$. Accordingly, we consider the polynomial

$$P_n(x) = x + a_2x^2 + \dots + a_nx^n \quad , \tag{1.22}$$

and following our general approach we define the function $s_n(x)$ by setting

$$S_n(x) = P_n(x) \quad \text{if} \quad 0 \leqq x < 1 \quad , \tag{1.23}$$

and extend its definition so as to satisfy

$$S_n(x) = xS_n(x+1), \ S_n(1) = 1, \quad \text{for all real} \quad x \quad . \tag{1.24}$$

We pass now to the critical continuity requirements on $S_n(x)$. Observe that by (1.23) and (1.24) we obtain the following:

$$\text{In} \quad 0 \leqq x < 1 \quad \text{we have} \quad S_n(x) = P_n(x) \quad , \tag{1.25}$$

$$\text{in} \quad -1 \leqq x < 0 \quad \text{we have} \quad S_n(x) = xS_n(x+1) = xP_n(x+1) \quad . \tag{1.26}$$

If we want these two functions to join at $x = 0$ with $n - 1$ continuous derivatives, we must ask that

$$P_n(x) - xP_n(x+1) = O(x^n), \quad \text{as} \quad x \to 0 \quad . \tag{1.27}$$

Our main result is

Theorem 2. For $n = 1,2,\dots$ there is a unique polynomial

$$P_n(x) = x + a_2^{(n)}x^2 + \dots + a_n^{(n)}x^n \tag{1.28}$$

satisfying (1.27) with $P_n(1) = 1$. Its coefficients are rational numbers.

A proof is given in §3. The first six polynomials of this remarkable sequence are

$$\begin{aligned}
P_1(x) &= P_2(x) = x \\
P_3(x) &= x + \frac{1}{2}x^2 - \frac{1}{2}x^3 \\
P_4(x) &= x + \frac{4}{7}x^2 - \frac{5}{7}x^3 + \frac{1}{7}x^4 \\
P_5(x) &= x + \frac{4}{7}x^2 - \frac{23}{35}x^3 + \frac{1}{35}x^4 + \frac{2}{35}x^5 \\
P_6(x) &= x + \frac{15}{26}x^2 - \frac{17}{26}x^3 - \frac{1}{26}x^4 + \frac{4}{26}x^5 - \frac{1}{26}x^6 \quad .
\end{aligned} \tag{1.29}$$

The question as to how the C^{n-1}-continuity of $S_n(x)$ at $x = 0$ is transmitted by (1.24) to the other integers is easily answered as follows.

(1.30) In $[-2,-1)$ $S_n(x) = xS_n(x+1) = x(x+1)S_n(x+2) = x(x+1)P_n(x+2)$.

From (1.27) we find that $P_n(x+1) - (x+1)P_n(x+2) = O(x+1)^n$ as $x \to -1$, whence

$$xP_n(x+1) - x(x+1)P_n(x+2) = Ox(x+1)^n = O(x+1)^n \quad \text{as} \quad x \to -1 \quad .$$

But then (1.26) and (1.30) show that

$$S_m^{(\nu)}(-1+0) = S_n^{(\nu)}(-1-0), \ (\nu = 0,\ldots,n-1) \quad ,$$

and we readily find similarly that

(1.31) $S_n^{(\nu)}(k+0) = S_n^{(\nu)}(k-0)$, $(\nu = 0,\ldots,n-1)$ <u>for all integers</u> $k < 0$.

Matters are slightly different if we pass to positive k.

(1.32) $$\text{In} \quad [1,2) : S_n(x) = \frac{S_n(x-1)}{x-1} = \frac{P_n(x-1)}{x-1} \quad .$$

Now (1.27) shows that $P_n(x-1) - (x-1)P_n(x) = O(x-1)^n$ and so

(1.33) $$\frac{P_n(x-1)}{x-1} - P_n(x) = O(x-1)^{n-1} \quad \text{as} \quad x \to 1 \quad .$$

But then (1.23) and (1.32) show that

$$S_n^{(\nu)}(1+0) = S_n^{(\nu)}(1-0) \quad \underline{\text{only for}} \quad \nu = 0,\ldots,n-2 \quad ,$$

so that <u>we have lost at</u> $x = 1$ <u>one order of continuity.</u>

From here on there is no further loss of continuity:

(1.34) $$\text{In} \quad [2,3) : S_n(x) = \frac{1}{x-1}\,\frac{S(x-2)}{x-2} = \frac{1}{x-1}\,\frac{P(x-2)}{x-2} \quad .$$

However, (1.33) gives $\dfrac{P_n(x-2)}{x-2} - P_n(x-1) = O(x-2)^{n-1}$ as $x \to 2$, whence

$$\frac{1}{x-1}\,\frac{P_n(x-2)}{x-2} - \frac{P_n(x-1)}{x-1} = O\frac{(x-2)^{n-1}}{x-1} = O(x-2)^{n-1} \quad \text{as} \quad x \to 2 \quad ,$$

and (1.32), (1.34) show that

$$S_n^{(\nu)}(2+0) = S_n^{(\nu)}(2-0), \ (\nu = 0,\ldots,n-2) \quad .$$

This continues for all positive k and we obtain

Corollary 1. The approximation $S_n(x)$ defined by (1.23), (1.24) and (1.27) has the following continuity properties:

(1.35) $$S_n(x) \in C^{n-1} \text{ at the points } x = 0,-1,-2,\ldots ,$$

(1.36) $$S_n(x) \in C^{n-2} \text{ at the points } x = 1,2,3,\ldots .$$

The only exception is $S_1(x)$ which is continuous on all of $\mathbb{R}$.

The exceptional case of $S_1(x)$ is due to $P_1(x) = x$ and the identity $x - x(x+1) = x^2 = 0x^2$ as $x \to 0$.

A further immediate consequence of (1.24) is

Corollary 2. The approximation $S_n(x)$ interpolates $1/\Gamma(x)$ at all integers, hence

(1.37) $$S_n(\nu) = \frac{1}{\Gamma(\nu)} \quad \text{for} \quad \nu \in \mathbb{Z} .$$

We have referred above to $S_n(x)$ as an approximation of $1/\Gamma(x)$. Is it? This essential point we can not settle and wish to state it as

Conjecture 1. The sequence of polynomials $P_n(x)$ of Theorem 2 satisfy

(1.38) $$\lim_{n\to\infty} P_n(x) = \frac{1}{\Gamma(x)} \quad \text{uniformly in every circle of } \mathbb{C} .$$

As an extremely weak substitute we establish

Theorem 3. For $n = 1,2,\ldots$ we have

(1.39) $$P_n(x) > 0 \quad \text{if} \quad 0 < x \leq 1 ,$$

(1.40) $$P_n'(x) > 0 \quad \text{if} \quad 0 \leq x \leq 1 .$$

Proof of (1.39): We apply the Budan-Fourier theorem in its classical formulation; see e.g. [8] where it is derived from the Descartes-Jacobi theorem by means of a special totally positive matrix; see also [5, 316-318]. Writing $P_n(x) = P(x)$ we use the notation

(1.41) $$V(x) = v(P(x),P'(x),\ldots,P^{(n)}(x))$$

for the number of changes of sign in the sequence as indicated. If $Z(0,1)$ = number of zeros of $P(x)$ in $(0,1)$, then the theorem states that

(1.42) $$Z(0,1) \leq V(1) - V(0) .$$

However, the relation (1.27) will provide much information concerning $V(0)$ and $V(1)$: Writing $F(x) = P(x) - xP(x+1)$ we find by Leibnitz's formula that $F^{(\nu)}(x) = P^{(\nu)}(x) - xP^{(\nu)}(x+1) - \nu P^{(\nu-1)}(x+1)$. Setting $x = 0$, (1.27) shows that

$$P^{(\nu)}(0) = \nu P^{(\nu-1)}(1), \quad (\nu = 1,2,\ldots,n-1) \quad , \tag{1.43}$$

and the right side of (1.42) promises to vanish. In fact from $P(x) = x + a_2x^2 + \ldots + a_nx^n$ and (1.43) we find that

$$V(0) = v(0,1,a_2,\ldots,a_{n-2}, a_{n-1}, a_n) \tag{1.44}$$

$$V(1) = v(1,a_2,a_3,\ldots,a_{n-1}, P^{(n-1)}(1), a_n) \quad . \tag{1.45}$$

We distinguish two cases.

1. $a_{n-1}a_n < 0$. In this case we see from (1.45) that $V(1) = V(0)$, whatever the sign of $P^{(n-1)}(1)$ might be.

2. $a_{n-1}a_n > 0$. In this case

$$P^{(n-1)}(x) = a_{n-1}(n-1)! + a_n n! x$$

shows that $P^{(n-1)}(1)$ has the common sign of a_{n-1} and a_n, so that again

$$V(1) = V(0) \quad .$$

Above we have assumed that $a_{n-1}a_n \neq 0$, but the last result is seen to hold in any case, whether one or both a_{n-1} and a_n should vanish.

We postpone a proof of (1.40) to §4 because it used results from § 3.

Using (1.24) as in our proof of Corollary 1 we obtain

Corollary 3. *We have*

$$S_n(x) > 0 \quad \textit{if} \quad x > 0 \quad , \tag{1.46}$$

and

$$S_n(x) < 0 \ \textit{in} \ (-1,0),\ S_n(x) > 0 \ \textit{in} \ (-2,-1),\ S_n(x) < 0 \ \textit{in} \ (-3,-2) \ \text{a.s.f.} \tag{1.47}$$

with alternating signs.

A curious negative consequence is

Corollary 4. The function $G(x) = \log S_n(x)$ is not a concave function for $x > 0$.

This follows from a theorem of Bohr, Mollerup and Artin [2, Theorem 2.1 on page 14] that $F(x) = \log \Gamma(x)$ is the only convex solution of the functional equation

$$F(x+1) - F(x) = \log x \quad \text{for} \quad x > 0, \quad \text{with} \quad F(1) = 0 \quad .$$

It suffices to observe that, by (1.24) and (1.46), the function $G(x) = \log S_n(x)$ satisfies

$$G(x+1) - G(x) = -\log x \quad \text{for} \quad x > 0, \quad \text{with} \quad G(1) = 0 \quad ,$$

while the identity $G(x) = -F(x)$ is evidently impossible.

Everybody believes the Euler constant

$$\gamma = \lim\left(\sum_1^n \frac{1}{\nu} - \log n\right) = .57721\ 56649 \ldots$$

to be irrational, but as far as we know nobody has proved this. We also believe it, and will here assume that

(1.48) $\qquad \gamma$ is an irrational number .

By (1.37) we know that $S_n(x)$ interpolates $\Gamma^{-1}(x)$ at all integers. However, much more is true as stated in

Corollary 5. We assume (1.48) to hold, and let $n \geq 3$. For each $k \in \mathbb{Z}$ the two open arcs

(1.49) $\qquad y = S_n(x)\ (k < x < k+1)$ and $y = \Gamma^{-1}(x)\ (k < x < k+1)$

cross each other.

Proof. Let $k = 0$. Assuming that the arcs (1.49) do not cross and let us get a contradiction. More precisely let us assume that

(1.50) $\qquad S_n(x) = P_n(x) \leq \Gamma^{-1}(x) \quad \text{if} \quad 0 < x < 1 \quad .$

From (1.32) we have

$$S_n(x) = \frac{S_n(x-1)}{x-1}, \quad \Gamma^{-1}(x) = \frac{\Gamma^{-1}(x-1)}{x-1} \quad \text{if} \quad 1 < x < 2 \quad ,$$

and therefore

$$\frac{S_n(x)}{\Gamma^{-1}(x)} = \frac{S_n(x-1)}{\Gamma^{-1}(x-1)} \leqq 1 \quad \text{by (1.50)} \quad .$$

Hence

$$S_n(x) \leqq \Gamma^{-1}(x) \quad \text{if} \quad 1 < x < 2 \quad . \tag{1.51}$$

However, (1.50) and (1.51) show that $S_n(x) \leqq \Gamma^{-1}(x)$ throughout the interval $0 < x < 2$. Since $S_n(1) = \Gamma^{-1}(1)$, while $S_n \in C^1(\mathbf{R})$, we conclude that their slopes at $x = 1$ must be equal. This is impossible by (1.48) because $(d/dx)\Gamma^{-1}(1) = -\Gamma'(1)/\Gamma(1)^2 = -\Gamma'(1) = \gamma$, while $S_n'(1)$ is evidently a rational number.

We illustrate numerically our conjecture (1.38) as well as Corollary 5 for the case when $n = 6$. With $P_6(x)$ as given in (1.29) and the table of $\Gamma(x)$ [1, Table 6.1 on page 267] we find that the functions $P_6(x)$ and $\Gamma^{-1}(x)$ cross at a point $\xi = .1182...$, and that we have

$$0 < \Gamma^{-1}(x) - P_6(x) < .000\ 001 \quad \text{if} \quad 0 < x < \xi \quad ,$$

$$0 < P_6(x) - \Gamma^{-1}(x) < .000\ 15 \quad \text{if} \quad \xi < x < 1 \quad .$$

From the evident proportion

$$\frac{S_n(x)}{\Gamma^{-1}(x)} = \frac{S_n(x+k)}{\Gamma^{-1}(x+k)}, \quad (x \notin \mathbf{Z},\ K \in \mathbf{Z}) \quad ,$$

it is clear, that for $n = 6$, the two arcs (1.49) cross at the point $x = \xi + k$.

Concluding we wish to point out that the two papers [4] and [11] deal with subjects related to the topic of the present note.

2. The equation $f(x+1) - f(x) = h(x)$: A proof of Theorem 1. We apply our previous method: Let us construct a compositie function $S_n(x)$ having the two properties:

(2.1) $S_n(x) = P_n(x) = \frac{1}{1!} a_1 x + \frac{1}{2!} a_2 x^2 + \ldots + \frac{1}{n!} a_n x^n$ if $0 \leqq x < 1$,

(2.2) $S_n(x)$ satisfies $S_n(x+1) - S_n(x) = h(x)$ for all real x .

On this function we now impose the continuity condition:

(2.3) $$S_n(x) \in C^{n-1}(\mathbf{R}) \quad .$$

We claim: If $S_n(x)$ is of class C^{n-1} in a neighborhood of $x = 0$ then (2.3) holds.

Proof. We write $S_n(x) = S(x)$. Differentiation of (2.2) shows that

$$S^{(\nu)}(k+1+0) - S^{(\nu)}(k+0) = h^{(\nu)}(k+0),$$
$$S^{(\nu)}(k+1-0) - S^{(\nu)}(k-0) = h^{(\nu)}(k-0)$$

and by subtraction

(2.4) $$S^{(\nu)}(k+1+0) - S^{(\nu)}(k+1-0) = S^{(\nu)}(k+0) - S^{(\nu)}(k-0) \quad .$$

This implies that if the right side of (2.4) vanishes for $k = 0$, it must vanish for all k and (2.3) is established.

To inforce C^{n-1}-continuity of S_n at $x = 0$ we observe the following: By (2.1) and (2.2) we have:

In $[0,1)$: $S(x) = P_n(x)$, while in $[-1,0)$: $S(x) = S(x+1) - h(x) = P_n(x+1) - h(x)$. Therefore the equations $S^{(\nu)}(+0) = S^{(\nu)}(-0)$, $(\nu = 0,\ldots,n-1)$ are equivalent to $P_n^{(\nu)}(0) = P_n^{(\nu)}(1) - h^{(\nu)}(0)$, and hence to

(2.5) $$P_n^{(\nu)}(1) = P_n^{(\nu)}(0) + h^{(\nu)}(0), \ (\nu = 0,\ldots,n-1) \quad .$$

The remainder of the proof is divided into several parts.

α) Construction of the $P_n(x)$. In terms of our explicit expresssion (2.1) of $P_n(x)$, the system (2.5) is seen to be

$$(2.6)\qquad \begin{aligned} \frac{1}{1!}a_1 + \frac{1}{2!}a_2 + \dots + \frac{1}{n!}a_n &= h(0) \\ \frac{1}{1!}a_2 + \dots + \frac{1}{(n-1)!}a_n &= h'(0) \\ \ddots \quad \frac{1}{1!}a_n &= h^{(n-1)}(0) \quad , \end{aligned}$$

and let us solve it for the a_ν.

The matrix of (2.6) is seen to be upper-triangular and also "striped", and the inversion of such a matrix is equivalent to the expansion of the reciprocal of a polynomial, as seen from the equivalence of the two relations

$$(2.7)\qquad \left\| \begin{matrix} C_1 & C_2 & \cdots & C_n \\ 0 & C_1 & \cdots & C_{n-1} \\ & & \ddots & \\ 0 & & & C_1 \end{matrix} \right\|^{-1} = \left\| \begin{matrix} D_1 & D_2 & \cdots & D_n \\ 0 & D_1 & \cdots & D_{n-1} \\ & & \ddots & \\ 0 & & & D_1 \end{matrix} \right\| ,$$

$$(2.8)\qquad \frac{1}{C_1 + C_2 t + \dots + C_n t^{n-1}} = D_1 + D_2 t + \dots + D_n t^{n-1} + O(t^n), \quad (\text{as} \quad t \to 0) \quad .$$

For the matrix of (2.6) we find that

$$\frac{1}{1!} + \frac{1}{2!}t + \dots + \frac{1}{n!}t^{n-1} = \frac{e^t - 1}{t} + O(t^n)$$

and therefore

$$1/(\frac{1}{1!} + \frac{1}{2!}t + \dots + \frac{1}{n!}t^{n-1}) = \frac{t}{e^t - 1} + O(t^n) = \sum_0^{n-1} \frac{B_\nu}{\nu!} + O(t^n)$$

where B_ν are the Bernoulli numbers. It follows that the solution of (2.6) is given by the matrix equation

$$
\begin{Vmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Vmatrix} = \begin{Vmatrix} 1 & \frac{B_1}{1!} & \cdots & \frac{B_{n-1}}{(n-1)!} \\ 0 & 1 & \cdots & \frac{B_{n-2}}{(n-2)!} \\ & & \ddots & \\ 0 & & & 1 \end{Vmatrix} \cdot \begin{Vmatrix} h(0) \\ h'(0) \\ \vdots \\ h^{(n-1)}(0) \end{Vmatrix}
$$

or

$$
a_k = \sum_{\nu=0}^{n-k} \frac{B_\nu}{\nu!} h^{(k+\nu-1)}(0)
$$

and so

$$
P_n(x) = \sum_{k=1}^{n} \frac{x^k}{k!} a_k = \sum_{k=1}^{n} \frac{x^k}{k!} \sum_{\nu=0}^{n-k} \frac{B_\nu}{\nu!} h^{(k-1+\nu)}(0) \quad .
$$

Replacing k by $k+1$ we find that

$$
P_n(x) = \sum_{k=0}^{n-1} \frac{x^{k+1}}{(k+1)!} A_{n,k} \tag{2.9}
$$

where

$$
A_{n,k} = \sum_{\nu=0}^{n-k-1} \frac{B_\nu}{\nu!} h^{(\nu+k)}(0) = \sum_{\nu=k}^{n-1} \frac{B_{\nu-k}}{(\nu-k)!} h^{(\nu)}(0) \quad . \tag{2.9'}
$$

β) <u>Construction of the solution</u> $f(x)$. Let m show that the polynomials (2.9) converge if $0 \leq x \leq 1$ and determine their limit. From $\limsup |B_\nu/\nu!|^{1/\nu} = 1/2\pi$ we conclude that for a small $\delta > 0$ we have $|B_\nu/\nu!|^{1/\nu} < 1/(2\pi-\delta)$ for sufficiently large ν, and so

$$
\left|\frac{B_\nu}{\nu!}\right| < C_1\left(\frac{1}{2\pi-\delta}\right)^\nu \quad \text{for all } \nu \quad .
$$

But then our assumption (1.11) shows that

$$
\left|\frac{B_\nu}{\nu!} h^{(\nu+k)}(0)\right| < C_2 A^k \left(\frac{A}{2\pi-\delta}\right)^\nu \quad \text{for all } \nu, \text{ with } C_2 \text{ constant} \quad . \tag{2.10}
$$

In view of (1.12) we may assume δ to be so chosen that $A < 2\pi - \delta$. Then (2.10) shows that S_k is well defined by

$$
S_k = \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} h^{(\nu+k)}(0) \tag{2.11}
$$

and that

(2.12) $$|S_k| < C_3 A^k \quad \text{for all } k,\ C_3 \text{ constant.}$$

But then

(2.13) $$f(x) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} S_k$$

is an entire function satisfying (1.13).

γ) Let us show that

(2.14) $$\lim_{n\to\infty} P_n(x) = f(x), \quad \underline{\text{uniformly in}} \quad 0 \leqq x \leqq 1 \ .$$

By (2.9) and (2.13)

$$f(x) - P_n(x) = \sum_{k=0}^{n-1} \frac{x^{k+1}}{(k+1)!}(S_k - A_{n,k}) + \sum_{k=n}^{\infty} \frac{x^{k+1}}{(k+1)!} S_k \ ,$$

and that it suffices to assume $0 \leqq x \leqq 1$, and to show that

$$R_n(x) = \sum_{k=0}^{n-1} \frac{x^{k+1}}{(k+1)!}(S_k - A_{n,k}) \to 0 \quad \text{as} \quad n \to \infty \ ,$$

uniformly in $[0,1]$.

In $[0,1]$ we have from (2.10) that

$$|R_n(x)| \leqq \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \sum_{\nu=n-k}^{\infty} \left|\frac{B_\nu}{\nu!} h^{(\nu+k)}(0)\right|$$

$$\leqq C_2 \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \sum_{\nu=n-k}^{\infty} \frac{A^{\nu+k}}{(2\pi-\delta)^{\nu}} \ ,$$

and replacing the index ν by $\nu+n-k$

$$|R_n(x)| \leqq C_2 \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \sum_{\nu=0}^{\infty} \frac{A^{\nu+n}}{(2\pi-\delta)^{n-k+\nu}}$$

$$\leqq C_2 \left(\frac{A}{2\pi-\delta}\right)^n \sum_{k=0}^{n-1} \sum_{\nu=0}^{\infty} \frac{(2\pi-\delta)^k}{(k+1)!} \left(\frac{A}{2\pi-\delta}\right)^{\nu}$$

$$\leqq C_2 \left(\frac{A}{2\pi-\delta}\right)^n \sum_{\nu=0}^{\infty} \sum_{k=0}^{\infty} \frac{(2\pi)^k}{(k+1)!} \left(\frac{A}{2\pi-\delta}\right)^{\nu} \leqq C_4 \left(\frac{A}{2\pi-\delta}\right)^n \to 0 \ ,$$

which establishes (2.14).

δ) Proof that

$$f(x+1) - f(x) = h(x) \quad . \tag{2.15}$$

Let

$$Q_n(x) = P_n(x+1) - P_n(x) \quad . \tag{2.16}$$

By (2.5) we have $Q_n^{(\nu)}(0) = P_n^{(\nu)}(1) - P_n^{(\nu)}(0) = h^{(\nu)}(0)$, which shows that

$$Q_n(x) = \sum_0^{n-1} \frac{x^\nu}{\nu!} h^{(\nu)}(0) \quad .$$

But then (2.16) becomes

$$P_n(x+1) - P_n(x) = \sum_0^{n-1} \frac{x^\nu}{\nu!} h^{(\nu)}(0)$$

and now (2.14) implies (2.15).

Remarks. 1. Is $f(x)$ the unique solution of (1.10) satisfying (1.13)? That it is we see as follows: The difference of two solutions of (1.10) is also entire of exponential type $< 2\pi$ and is also periodic of period 1. By a general theorem (see [3, Theorem 6.10.1 on page 109]) this difference must reduce to a constant.

2. Since $S_n(x)$ and $f(x)$ satisfy (1.15) and (1.10) we have

$$S_n(x+1) - S_n(x) - (f(x+1) - f(x)) = h(x) - h(x) = 0$$

and so

$$S_n(x+1) - f(x+1) = S_n(x) - f(x) \quad \text{for all } x \quad .$$

This shows that the uniform convergence in (2.14) implies that

$$\lim_{n\to\infty} S_n(x) = f(x) \quad \underline{\text{uniformly for all real}} \quad x \quad . \tag{2.17}$$

3. <u>The equation</u> $f(x) = xf(x+1)$: <u>A proof of Theorem</u> 2. The matter seems quite simple if we use the proper tools. From (1.27) for $x = 0$ we get that $P_n(0) = 0$. For $P(x) = x$ the left side of (1.27) becomes $x - x(x+1) = -x^2$, and this shows that $P_1(x) = P_2(x) = x$.

Assuming $n > 2$, (1.27) gives $P_n'(x) - P_n(x+1) - xP_n'(x+1) = Onx^{n-1}$, and for $x = 0$ that $P_n'(0) = P_n(1) = 1$. For

$$P_n(x) = x + a_2x^2 + \dots + a_nx^n \tag{3.1}$$

we write (1.27) explicitly and find

$$\begin{aligned}
&x + a_2x^2 + a_3x^3 + \dots + a_nx^n \\
&- x(x+1) \\
&- a_2x(x^2 + 2x+1) \\
&- a_3x(x^3 + 3x^2 + 3x + 1) \\
&\quad\vdots \\
&- a_nx\left(x^n + \tbinom{n}{1}x^{n-1} + \tbinom{n}{2}x^{n-2} + \dots + 1\right) = Ox^n \quad \text{as} \quad x \to 0 \ .
\end{aligned}$$

Collecting terms and writing that the coefficients of $x, x^2, \dots, x^{n-1}$, vanish, we get the equations

$$\begin{array}{llllllll}
 & & -a_2 - a_3 & - \dots - a_{n-2} & - a_{n-1} & - a_n & = 0 \\
a_2 & -1 & -2a_2 - 3a_3 & - \dots - \binom{n-2}{1}a_{n-2} & - \binom{n-1}{1}a_{n-1} & - \binom{n}{1}a_n & = 0 \\
a_3 & & -a_2 - 3a_3 & - \dots - \binom{n-2}{2}a_{n-2} & - \binom{n-1}{2}a_{n-1} & - \binom{n}{2}a_n & = 0 \\
a_4 & & \quad\ - a_3 & - \dots - \binom{n-2}{3}a_{n-2} & - \binom{n-1}{3}a_{n-1} & - \binom{n}{3}a_n & = 0 \\
\vdots & & & & & & \\
a_{n-1} & & & - a_{n-2} & - \binom{n-1}{n-2}a_{n-1} & - \binom{n}{n-2}a_n & = 0 \ .
\end{array}$$

This is a system of $n - 1$ equations in $a_2, \dots, a_n$. Rearranging these equations we may write

$$
(3.2)\quad
\begin{array}{lllllllr}
a_2 + & a_3 + & a_4 & +\cdots+ & a_{n-2} + & a_{n-1} + & a_n & = 0 \\
(2-1)a_2 + & 3a_3 + & 4a_4 & +\cdots+ & \binom{n-2}{1}a_{n-2} + & \binom{n-1}{1}a_{n-1} + & \binom{n}{1}a_n & = -1 \\
a_2 & +(3-1)a_3 + & 6a_4 & +\cdots+ & \binom{n-2}{2}a_{n-2} + & \binom{n-1}{2}a_{n-1} + & \binom{n}{2}a_n & = 0 \\
 & a_3 & +(4-1)a_4 & +\cdots+ & \binom{n-2}{3}a_{n-2} + & \binom{n-1}{3}a_{n-1} + & \binom{n}{3}a_n & = 0 \\
 & & \cdot & & & & & \\
 & & & \cdot & & & & \\
 & & & & a_{n-2} + & \left\{\binom{n-1}{n-2} - 1\right\}a_{n-1} + & \binom{n}{n-2}a_n & = 0 \;.
\end{array}
$$

Observe the simple structure of the determinant Δ_n of this system. For instance

$$
\Delta_5 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2-1 & 3 & 4 & 5 \\ 1 & 3-1 & 6 & 10 \\ 0 & 1 & 4-1 & 10 \end{vmatrix} .
$$

It is obtained from a solid minor

$$
D_5 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 \\ 0 & 1 & 4 & 10 \end{vmatrix}
$$

of the Pascal triangle by subtracting 1 from the elements of the diagonal just below its main diagonal. A simple induction shows that all these minors are $= 1$:

$$(3.3)\qquad D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 2 & 3 & & \binom{n}{1} \\ 1 & 3 & & \cdot \\ & 1 & & \cdot \\ & & \ddots & \vdots \\ & & 1 \;\cdots & \binom{n}{n-2} \end{vmatrix} = 1 \; .$$

Indeed by successively subtracting each column from the next one we find

$$D_5 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 6 \end{vmatrix} = D_4 \quad \text{a.s.f.}$$

We now consider the determinant of the system (3.2)

$$(3.4)\qquad \Delta_n = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2-1 & 3 & \binom{n-2}{1} & \binom{n-1}{1} & \binom{n}{1} \\ 1 & 3-1 & & & \cdot \\ 0 & 1 & & & \cdot \\ & & \ddots & & \\ 0 & & 1 & \binom{n-1}{n-2}-1 & \binom{n}{n-2} \end{vmatrix}$$

and wish to prove

<u>Lemma</u> 1. <u>We have</u>

$$(3.5)\qquad \Delta_n \geq 1 \quad \underline{\text{for}} \quad n = 1,2,\ldots \; .$$

Proof. 1°. We use the fact first pointed out in [7] (see also [8]) that the infinite Pascal triangle is totally positive, i.e. all its minors, of all orders, are ≥ 0.

2°. We split the first $n - 2$ columns of (3.4) into two columns, the second columns containing only vanishing elements, except the single element $= -1$. In this way Δ_n as a sum of 2^{n-2} determinants

$$(3.6) \qquad \Delta_n = D_n + \sum_{r \geq 1} \Delta(i_1, i_2, \ldots, i_r) \quad , \; (1 \leq i_1 < i_2 < \ldots < i_r \leq n-2) \; .$$

where $D_n = 1$, by (3.3), while $(i_1, \ldots, i_r)$ runs through all combination of r among the numbers $1, 2, \ldots, n-2$, $(r>1)$. Thus $\Delta(i_1, \ldots, i_r)$ is the determinant obtained from D_n by replacing its columns $i_1, i_2, \ldots, i_r$ by the columns

$$(0, \ldots, 0, \overset{(i_1+1)}{-1}, 0, \ldots, 0)^T, \ldots, (0, \ldots, 0, \overset{(i_r+1)}{-1}, 0, \ldots, 0)^T \; .$$

We summarize its structure by

$$(3.7) \qquad \Delta(i_1, \ldots, i_r) = \begin{array}{c} \begin{array}{ccc} (i_1) & (i_2) & (i_r) \end{array} \\ \left| \begin{array}{ccc} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots \\ -1 & \vdots & \vdots \\ 0 & 0 & \vdots \\ \vdots & -1 & \vdots \\ \vdots & 0 & 0 \\ \vdots & \vdots & -1 \\ \vdots & \vdots & 0 \\ 0 & 0 & 0 \end{array} \right| \end{array}$$

Here all elements are the old elements of D_n except for those in the r columns $i_1, i_2, \ldots, i_r$.

3°. We apply Laplace's rule (see [5, page 6]) of expanding the determinant (3.7) by all its minors of order r, from its r columns $i_1, \ldots, i_r$, multiplied by their algebraic compliments. This expansion reduces evidently to a single term

$$(3.8) \qquad \Delta(i_1,\dots,i_r) = \begin{vmatrix} -1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{vmatrix}_{r\times r} \times C ,$$

where

$$(3.9) \qquad C = (-1)^{\sum_{j=1}^{r} (i_j+1) + \sum_{j=1}^{r} i_j} \times \mathcal{D} ,$$

where $\mathcal{D}$ is a minor of D_n, and therefore $\mathcal{D} \geqq 0$ by 1°.

4°. From (3.8) and (3.9) we find that

$$(3.10) \qquad \Delta(i_1,\dots,i_r) = (-1)^r(-1)^{2\sum i_j+r} \cdot \mathcal{D} = \mathcal{D} \geqq 0 .$$

Now (3.3), (3.6) and (3.10) show that (3.5) is established.

Remark. Elimination of the unknowns $a_2,\dots,a_n$ between (3.1) and the system (3.2) allows us to write $P_n(x)$ explicitly in terms of a quotient of two determinants, e.g.

$$P_5(x) = x + \begin{vmatrix} x^2 & x^3 & x^4 & x^5 \\ 1 & 1 & 1 & 1 \\ 1 & 3-1 & 6 & 10 \\ 0 & 1 & 4-1 & 10 \end{vmatrix} : \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2-1 & 3 & 4 & 5 \\ 1 & 3-1 & 6 & 10 \\ 0 & 1 & 4-1 & 10 \end{vmatrix} ,$$

which is easily seen to agree with its expression as given in (1.29). The above expression of $P_5(x)$ is readily continued for $n > 5$ by simply adding obvious last rows and columns to both determinants; the only trouble is that we obtain $P_n(x) = x +$ a ratio of two determinants of order $n - 1$.

4. A proof of the second part (1.40) of Theorem 3. We know that the coefficients $a_2,\ldots,a_n$ of the polynomial (3.1) are the solutions of the system of equations (3.2). Let us use this fact to show first that

$$a_2 = \frac{1}{2} P_n''(0) > 0 \quad . \tag{4.1}$$

By Cramer's rule

$$a_2 = \tilde{\Delta}_n/\Delta_n \quad , \tag{4.2}$$

where Δ_n is given by (3.4), and $\tilde{\Delta}_n$ is obtained from Δ_n by omitting its first column and second row. We know by (3.5) that $\Delta_n \geqq 1$. Let us show that also

$$\tilde{\Delta}_n \geqq 1 \quad . \tag{4.3}$$

This is done by applying to the determinant $\tilde{\Delta}_n$ precisely the procedure previously applied to Δ_n to prove (3.5). Here we need the following stronger form of the total positivity of the matrix of the determinant D_n of (3.3):

(4.4) A minor of D_n which doe not vanish formally, i.e. because it has too many zero elements, is positive.

Denote by $\tilde{D}_n$ the minor of D_n obtained by omitting its first column and second row. This minor having an integer value, the property (4.4) implies that

$$\tilde{D}_n \geqq 1 \quad . \tag{4.5}$$

From this point on the proof of (3.5) as given in §3 applied without any essential change to establish (4.3). Now (3.5) and (4.2) imply (4.1).

To establish (1.40) we again apply the Budan-Fourier theorem, this time to $P_n'(x)$. Let Z' denote the number of zeros of $P_n'(x)$ in $0 < x < 1$. Writing $V'(x) = v(P_n'(x), P_n''(x),\ldots,P_n^{(n)}(x))$ we obtain by that theorem that

$$Z' \leqq V'(1) - V'(0) \quad , \tag{4.6}$$

while the equations (1.43) show that

$$V'(0) = v(1, a_2,\ldots,a_{n-2}, a_{n-1}, a_n)$$

$$V'(1) = v(a_2,a_3,\ldots,a_{n-1}, P_n^{(n-1)}(1), a_n) \quad .$$

Since $a_2 > 0$ by (4.1), it follows as in our proof of (1.39), that the right side of (4.6) vanishes, hence $Z' = 0$. Since $P_n'(0) = 1$, we have established (1.40).

5. <u>On the nature of our Conjecture 1 of (1.38).</u> The coefficients C_k of the expansion

$$\frac{1}{\Gamma(x)} = \sum_{k=1}^{\infty} c_k x^k, \quad (c_1 = 1) \tag{5.1}$$

are given in [1, page 256] to 15 decimal places, as originally computed by H. T. Davis in 1933, with corrections due to H. E. Salzer. If we substitute the expansion (5.1) of $f(x) = 1/\Gamma(x)$ into the functional equation $f(x) = xf(x+1)$ we find that the c_k satisfy an infinite system of linear equations obtained from our system (3.2) by substituting c_k for a_k and then letting $n \to \infty$.

The problem is to show that the solutions $a_k^{(n)}$ $(k = 2,\ldots,n)$ of the partial system (3.2) converge element-wise to the solutions c_k of the infinite system. More specifically we wish to show that

$$P_n(x) = \sum_{k=1}^{n} a_k^{(n)} x^k \to \sum_{k=1}^{\infty} c_k x^k \quad ,$$

uniformly in every circle of $\mathbb{C}$.

In 1913 F. Riesz devoted a book [6] to such problems which he calls "Problèmes des réduites". His general theory does not apply to our specific problem. However, further work in Functional Analysis might solve it. This shows that Eulerian mathematics still presents to contemporary analysts challenging problems.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Our title describes a phenomenon best illustrated by our Theorem 1.

1. Let $h(x)$ be an entire function of exponential type $A < 2\pi$. We show that there is a unique entire function $f(x)$ of exponential type A satisfying the functional equation

$$(1) \qquad f(x+1) - f(x) = h(x), \quad \text{with} \quad f(0) = 0 \ .$$

2. We define a function $S_n(x)$ which reduces to a polynomial in $[0,1)$:

(continued)

ABSTRACT (continued)

(2) $$S_n(x) = a_1x + a_2x^2 + \ldots + a_nx^n \quad \text{if} \quad 0 \leqq x < 1 \ ,$$

and we extend the definition of $S_n(x)$ to all real x by asking that it satisfy

(3) $$S_n(x+1) - S_n(x) = h(x) \quad \text{for all real} \quad x.$$

If we require also that

(4) $$S_n^{(\nu)}(+0) = S_n^{(\nu)}(-0) \quad \text{for} \quad \nu = 0,1,\ldots,n-1$$

then we have

(5) $$S_n(x) \in C^{n-1}(\mathbb{R})$$

and also

(6) $$\lim_{n\to\infty} S_n(x) = f(x) \quad \text{uniformly for all real} \quad x \ .$$

Our title stresses the phenomenon that the C^{n-1}-continuity requirement (5) implies the approximation property (6).

As a second example we deal with the functional equation

(7) $$f(x) = xf(x+1), \quad \text{with} \quad f(1) = 1 \ ,$$

satisfied by $f(x) = 1/\Gamma(x)$.